

SOME APPROXIMATION RESULTS ON HIGHER ORDER GENERALIZATION OF BERNSTEIN TYPE OPERATORS DEFINED BY (p, q) -INTEGERS

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ABSTRACT. In this paper, we introduce the higher order generalization of Bernstein type operators defined by (p, q) -integers. We establish some approximation results for these new operators by using the modulus of continuity.

1. INTRODUCTION AND PRELIMINARIES

In 1912, S.N Bernstein [2] introduced the following sequence of operators $B_n : C[0, 1] \rightarrow C[0, 1]$ defined for any $n \in \mathbb{N}$ and for any $f \in C[0, 1]$ such as

$$B_n(f; x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0, 1]. \quad (1.1)$$

In approximation theory, q -type generalization of Bernstein polynomials was introduced by Lupaş [5].

For $f \in C[0, 1]$, the generalized Bernstein polynomial based on the q -integers is defined by Phillips [10] as follows

$$B_{n,q}(f; x) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x) f\left(\frac{[k]_q}{[n]_q}\right), \quad x \in [0, 1]. \quad (1.2)$$

Recently, Mursaleen et al. [7] applied (p, q) -calculus in approximation theory and introduced first (p, q) -analogue of Bernstein operators(Revised) and defined as:

$$B_{n,p,q}(f; x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n f\left(\frac{[k]}{p^{k-n}[n]}\right) P_{n,k}(p, q; x), \quad 0 < q < p \leq 1, \quad x \in [0, 1] \quad (1.3)$$

where

$$P_{n,k}(p, q; x) = p^{\frac{k(k-1)}{2}} \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x). \quad (1.4)$$

They have also introduced and studied approximation properties based on (p, q) -integers given as: Bernstein-Stancu operators [8] and Bernstein-Shurer operators [9].

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We recall some basic properties of (p, q) -integers.

The (p, q) -integer $[n]_{p,q}$ is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \dots, \quad 0 < q < p \leq 1.$$

The (p, q) -Binomial expansion is

$$(x + y)_{p,q}^n := (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y)$$

and the (p, q) -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!}.$$

For $p = 1$, all the notions of (p, q) -calculus are reduced to q -calculus. For details on q -calculus and (p, q) -calculus, one can refer [11, 12, 1, 3] and [5], respectively. In this paper we use the notation $[n]$ in place of $[n]_{p,q}$.

In [3], (p, q) -derivative of a function $f(x)$ is defined by

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0, \quad (1.5)$$

and the formulae for the (p, q) -derivative for the product of two functions is given as

$$D_{p,q}(fg)(x) = f(px).D_{p,q}g(x) + \{D_{p,q}f(x)\}.g(qx), \quad (1.6)$$

also

$$D_{p,q}(fg)(x) = f(qx).D_{p,q}g(x) + \{D_{p,q}f(x)\}.g(px). \quad (1.7)$$

Let $r \in \mathbb{N} \cup \{0\}$ be a fixed number. For $f \in C^r[0, 1]$ and $m \in \mathbb{N}$, we define an operator of r^{th} order for (p, q) - Bernstein type operators as follows:

$$B_{n,p,q}^{[r]}(f; x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n P_{n,k}(p, q; x) \sum_{i=0}^r \frac{1}{i!} f^{(i)} \left(\frac{[k]}{p^{k-n}[n]} \right) \left(x - \frac{[k]}{p^{k-n}[n]} \right)^i. \quad (1.8)$$

In this paper, using the moment estimates from [6], we give the estimates of the central moments for operators defined by (1.3). We also study some approximation properties of an r^{th} order generalization of the operators defined by (1.8) using the techniques of the work on the higher order generalization of q -analogue [13]. Further, we study approximation properties and prove Voronovskaja type theorem for these operators.

If we put $p = 1$, then we get the moments for q -Bernstein operators [6] and the usual generalization higher order q -Bernstein operators [13], respectively.

2. MAIN RESULTS

The following result is (p, q) -analogue of [1].

Proposition 2.1. For $n \geq 1$, $0 < q < p \leq 1$

$$D_{p,q}(1+x)_{p,q}^n = [n](1+qx)_{p,q}^{n-1}. \quad (2.1)$$

Proof. By applying simple calculation on (p, q) -analogue, we have

$$(1+px)_{p,q}^n = p^{n-1}(1+px)(1+qx)_{p,q}^{n-1}, (1+qx)_{p,q}^n = (p^{n-1} + q^n x)(1+qx)_{p,q}^{n-1}. \quad (2.2)$$

Applying (p, q) -derivative and result (2.2) we get the desired result. \square

Lemma 2.2. Let $B_{n,p,q}(f; x)$ be given by (1.3). Then for any $m \in \mathbb{N}$, $x \in [0, 1]$ and $0 < q < p \leq 1$ we have

$$\begin{aligned} B_{n,p,q}((t-x)_{p,q}^{m+1}; x) &= \frac{p^{m+n}x(1-x)}{[n]} D_{p,q} \left\{ B_{n,p,q} \left((t - \frac{x}{p})_{p,q}^m; \frac{x}{p} \right) \right\} \\ &+ \frac{p^{m+n-1}[m]x(1-x)}{[n]} B_{n,p,q} \left((t - \frac{qx}{p})_{p,q}^{m-1}; \frac{qx}{p} \right) \\ &+ \frac{[m](p^n - q^n)x}{[n]} B_{n,p,q}((t-x)_{p,q}^m; x). \end{aligned}$$

Proof. First of all by using (1.6) and Lemma 2.1, we have

$$\begin{aligned} &D_{p,q} \left(\frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n \left(t - \frac{x}{p} \right)_{p,q}^m P_{n,k}(p, q; \frac{x}{p}) \right) \\ &= \frac{1}{p^{\frac{n(n-1)}{2}}} \left(\sum_{k=0}^n (t-x)_{p,q}^m D_{p,q} \{ P_{n,k}(p, q; \frac{x}{p}) \} - \frac{[m]}{p} \sum_{k=0}^n \left(t - \frac{qx}{p} \right)_{p,q}^{m-1} P_{n,k}(p, q; \frac{qx}{p}) \right). \end{aligned} \quad (2.3)$$

Now in the same way by using (1.6) and Lemma 2.1, we have

$$\begin{aligned} &D_{p,q} \left\{ P_{n,k} \left(p, q; \frac{x}{p} \right) \right\} = D_{p,q} \left\{ p^{\frac{k(k-1)}{2}} [k] \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left(\frac{x}{p} \right)^k \left(1 - \frac{x}{p} \right)^k \right\} \\ &= p^{\frac{k(k-1)}{2}} \left([k] \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{1}{p^k} x^{k-1} \left(1 - \frac{qx}{p} \right)_{p,q}^{n-k} - [n-k] \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{1}{p} x^k \left(1 - \frac{qx}{p} \right)_{p,q}^{n-k-1} \right). \end{aligned} \quad (2.4)$$

Now by a simple calculation, we have

$$\left(1 - \frac{qx}{p} \right)_{p,q}^{n-k} = \frac{1}{p^{n-k}} (p - qx)_{p,q}^{n-k+1} = \frac{1}{p^{n-k}} \frac{1}{(1-x)} (p^{n-k} - q^{n-k}x)(1-x)_{p,q}^{n-k} \quad (2.5)$$

$$\left(1 - \frac{qx}{p} \right)_{p,q}^{n-k-1} = \frac{1}{p^{n-k-1}} \frac{1}{(1-x)} (1-x)_{p,q}^{n-k}. \quad (2.6)$$

From (2.4), (2.5) and (2.6), we get

$$D_{p,q} \left\{ P_{n,k} \left(p, q; \frac{x}{p} \right) \right\} = \frac{P_{n,k}(p, q; x)}{p^n x(1-x)} ([k](p^{n-k} - q^{n-k}x) - p^k[n-k]x),$$

which implies that

$$D_{p,q} \left\{ P_{n,k} \left(p, q; \frac{x}{p} \right) \right\} = \frac{P_{n,k}(p, q; x)}{p^n x(1-x)} (p^{n-k}[k] - [n]x). \quad (2.7)$$

From (2.3), (2.7), we have

$$\begin{aligned} D_{p,q} \left(\sum_{k=0}^n \left(t - \frac{x}{p} \right)_{p,q}^m P_{n,k}(p, q; \frac{x}{p}) \right) \\ = -\frac{1}{p^{\frac{n(n-1)}{2}}} \frac{[m]}{p} \sum_{k=0}^n \left(t - \frac{qx}{p} \right)_{p,q}^{m-1} P_{n,k}(p, q; \frac{qx}{p}) \\ + \frac{1}{p^{\frac{n(n-1)}{2}}} \frac{1}{p^n x(1-x)} \sum_{k=0}^n (t-x)_{p,q}^m P_{n,k}(p, q; x) (p^{n-k}[k] - [n]x) \\ = -\frac{1}{p^{\frac{n(n-1)}{2}}} \frac{[m]}{p} \sum_{k=0}^n \left(t - \frac{qx}{p} \right)_{p,q}^{m-1} P_{n,k}(p, q; \frac{qx}{p}) \\ + \frac{1}{p^{\frac{n(n-1)}{2}}} \frac{1}{p^n x(1-x)} \sum_{k=0}^n (t-x)_{p,q}^m P_{n,k}(p, q; x) \\ \times \left(\frac{[n]}{p^m} (p^m t - q^m x) - \frac{[n]}{p^m} (p^m - q^m) x \right). \end{aligned}$$

Hence we have

$$\begin{aligned} D_{p,q} \left\{ B_{n,p,q} \left(\left(t - \frac{x}{p} \right)_{p,q}^m; \frac{x}{p} \right) \right\} \\ = -\frac{[m]}{p} B_{n,p,q} \left(\left(t - \frac{qx}{p} \right)_{p,q}^{m-1}; \frac{qx}{p} \right) + \frac{[n]}{p^{m+n} x(1-x)} B_{n,p,q} \left((t-x)_{p,q}^{m+1}; x \right) \\ - \frac{[m](p^n - q^n)}{p^{m+n}(1-x)} B_{n,p,q} \left((t-x)_{p,q}^m; x \right). \end{aligned}$$

This complete the proof of Lemma 2.2. \square

Lemma 2.3. Let $B_{n,p,q}((t-x)_{p,q}^m; x)$ be a polynomial in x of degree less than or equal to m and the minimum degree of $\frac{1}{[n]}$ is $\lfloor \frac{m+1}{2} \rfloor$. Then for any fixed $m \in \mathbb{N}$ and $x \in [0, 1]$, $0 < q < p \leq 1$ we have

$$B_{n,p,q}((t-x)_{p,q}^m; x) = \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}} \sum_{k=0}^{m-2} b_{k,m,n}(p, q) x^k, \quad (2.8)$$

such that the coefficients $b_{k,m,n}(p, q)$ satisfy $|b_{k,m,n}(p, q)| \leq b_m$, $k = 1, 2, \dots, m-2$ and b_m does not depend on x, t, p, q ; where $\lfloor a \rfloor$ is an integer part of $a \geq 0$.

Proof. By induction it is true for $m = 2$. Assuming it is true for m , then from Lemma 2.2 and equation (2.8) we have

$$B_{n,p,q}((t-x)_{p,q}^{m+1}; x)$$

$$\begin{aligned}
&= \frac{p^{m+n-1}x(1-x)}{[n]^{1+\lfloor \frac{m+1}{2} \rfloor}} D_{p,q} \left\{ x \left(1 - \frac{x}{p}\right) \sum_{k=0}^{m-2} b_{k,m,n}(p, q) \left(\frac{x}{p}\right)^k \right\} \\
&+ \frac{p^{m+n-1}[m]x(1-x)}{[n]^{1+\lfloor \frac{m}{2} \rfloor}} \sum_{k=0}^{m-3} b_{k,m-1,n}(p, q) \left(\left(\frac{q}{p}\right)^{k+1} x^{k+1} - \left(\frac{q}{p}\right)^{k+2} x^{k+2} \right) \\
&+ \frac{[m](p^n - q^n)x(1-x)}{[n]^{1+\lfloor \frac{m+1}{2} \rfloor}} \sum_{k=0}^{m-2} b_{k,m,n}(p, q) x^{k+1} \\
&= \frac{p^{m+n}x(1-x)}{[n]^{1+\lfloor \frac{m+1}{2} \rfloor}} \sum_{k=0}^{m-2} [k] b_{k,m,n}(p, q) \frac{1}{p^k} (x^k - x^{k+1}) \\
&+ \frac{p^{m+n-1}x(1-x)}{[n]^{1+\lfloor \frac{m+1}{2} \rfloor}} \sum_{k=0}^{m-2} b_{k,m,n}(p, q) \left(\frac{q}{p}\right)^k \left(x^k - \frac{[2]}{p} x^{k+1} \right) \\
&+ \frac{p^{m+n-1}[m]x(1-x)}{[n]^{1+\lfloor \frac{m}{2} \rfloor}} \sum_{k=0}^{m-3} b_{k,m-1,n}(p, q) \left(\left(\frac{q}{p}\right)^{k+1} x^{k+1} - \left(\frac{q}{p}\right)^{k+2} x^{k+2} \right) \\
&+ \frac{[m](p^n - q^n)x(1-x)}{[n]^{1+\lfloor \frac{m+1}{2} \rfloor}} \sum_{k=0}^{m-2} b_{k,m,n}(p, q) x^{k+1} \\
&= \frac{x(1-x)}{[n]^{1+\lfloor \frac{m+1}{2} \rfloor}} \sum_{k=0}^{m-2} (p^{m+n-k}[k] + p^{m+n-k-1}q^k) b_{k,m,n}(p, q) x^k \\
&- \frac{x(1-x)}{[n]^{1+\lfloor \frac{m+1}{2} \rfloor}} \sum_{k=1}^{m-1} (p^{m+n+1-k}[k-1]_{p,q} + [2]p^{m+n-k-1}q^{k-1}) b_{k-1,m,n}(p, q) x^k \\
&+ \frac{x(1-x)}{[n]^{1+\lfloor \frac{m}{2} \rfloor}} \sum_{k=1}^{m-2} [m]p^{m+n-k-1}q^k b_{k-1,m-1,n}(p, q) x^k \\
&- \frac{x(1-x)}{[n]^{1+\lfloor \frac{m}{2} \rfloor}} \sum_{k=2}^{m-1} [m]p^{m+n-k-1}q^k b_{k-2,m-1,n}(p, q) x^k \\
&+ \frac{x(1-x)}{[n]^{1+\lfloor \frac{m+1}{2} \rfloor}} \sum_{k=1}^{m-1} [m](p^n - q^n) b_{k-1,m,n}(p, q) x^k \\
&= \frac{x(1-x)}{[n]^{\lfloor \frac{m+2}{2} \rfloor}} \sum_{k=0}^{m-1} b_{k,m+1,n}(p, q) x^k
\end{aligned}$$

where

$$\begin{aligned}
b_{k,m+1,n}(p, q) &= \frac{1}{[n]^\alpha} (p^{m+n-k}[k] + p^{m+n-k-1}q^k) b_{k,m,n}(p, q) \\
&- \frac{1}{[n]^\alpha} (p^{m+n+1-k}[k-1] + [2]p^{m+n-k-1}q^{k-1}) b_{k-1,m,n}(p, q) \\
&+ \frac{1}{[n]^\alpha} [m](p^n - q^n) b_{k-1,m,n}(p, q) + [m]p^{m+n-k-1}q^k b_{k-1,m-1,n}(p, q) \\
&- [m]p^{m+n-k-1}q^k b_{k-2,m-1,n}(p, q).
\end{aligned}$$

Clearly

$$\alpha = 1 + \lfloor \frac{m+1}{2} \rfloor - \lfloor \frac{m+2}{2} \rfloor, \quad 0 \leq k \leq m-1,$$

which lead us that either $\alpha = 0$ or $\alpha = 1$.

Since $|b_{k,m,n}(p, q)| \leq b_m$, for $k = m-1$, clearly we have

$$\begin{aligned}
|b_{k,m+1,n}(p, q)| &\leq \frac{1}{[n]^\alpha} (p^{n+1}[m-1] + p^n q^{m-1}) b_m + \frac{1}{[n]^\alpha} (p^{n+2}[m-2] + [2]p^n q^{m-2}) b_m \\
&+ \frac{1}{[n]^\alpha} [m](p^n - q^n) b_m + [m]p^n q^{m-1} b_{m-1} \\
&+ [m]p^n q^{m-1} b_{m-1} \\
&= \frac{1}{[n]^\alpha} (p[m-1] + q^{m-1}) b_m + \frac{1}{[n]^\alpha} (p^2[m-2] + [2]q^{m-2}) b_m \\
&+ \frac{1}{[n]^\alpha} [m]b_m + [m]q^{m-1} b_{m-1} + [m]q^{m-1} b_{m-1} \\
&= b_{m+1}, \quad k = 1, 2, \dots, m-1,
\end{aligned}$$

and b_m does not depend on x, t, p, q . This complete the proof. \square

From the Lemma 2.2 and Lemma 2.3 we have the following theorem.

Theorem 2.4. *Let $m \in \mathbb{N}$ and $0 < q < p \leq 1$. Then there exists a constant $C_m > 0$ such that for any $x \in [0, 1]$, we have*

$$|B_{n,p,q}((t-x)_{p,q}^m; x)| \leq C_m \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}}.$$

Corollary 2.5. *Let $m \in \mathbb{N}$ and $0 < q < p \leq 1$. Then there exists a constant $K_m > 0$ such that for any $x \in [0, 1]$, we have*

$$B_{n,p,q}(|t-x|_{p,q}^m; x) \leq K_m \frac{x(1-x)}{[n]^{\frac{m}{2}}}. \quad (2.9)$$

Proof. For an even m , clearly we have

$$\begin{aligned} B_{n,p,q}(|t-x|_{p,q}^m; x) &= B_{n,p,q}((t-x)_{p,q}^m; x) \\ &\leq C_m \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}} \\ &= K_m \frac{x(1-x)}{[n]^{\frac{m}{2}}} \end{aligned}$$

In case if m is odd, say $m = 2u + 1$, we have

$$\begin{aligned} B_{n,p,q}(|t-x|_{p,q}^{2u+1}; x) &\leq \sqrt{B_{n,p,q}(|t-x|_{p,q}^{4u}; x)} \sqrt{B_{n,p,q}(|t-x|_{p,q}^2; x)} \\ &\leq \sqrt{C_{4u} \frac{x(1-x)}{[n]^{\lfloor \frac{4u+1}{2} \rfloor}}} \sqrt{C_2 \frac{x(1-x)}{[n]^{\lfloor \frac{3}{2} \rfloor}}} \\ &= \sqrt{C_{4u} \frac{x(1-x)}{[n]^{\frac{2u}{2}}}} \sqrt{C_2 \frac{x(1-x)}{[n]}} \\ &= K_{2u+1} \frac{x(1-x)}{[n]^{\frac{2u+1}{2}}}. \end{aligned}$$

This complete the proof. \square

Theorem 2.6. Let $B_{n,p,q}^{[r]}(f; x)$ be an operator from $C^r[0, 1] \rightarrow C^r[0, 1]$. Then for $0 < q < p \leq 1$ there exists a constant $M(r)$ such that for every $f \in C^r[0, 1]$, we have

$$\| B_{n,p,q}^{[r]}(f; x) \|_{C[0,1]} \leq M(r) \sum_{i=0}^r \| f^{(i)} \| = M(r) \| f \|_{C^r[0,1]}. \quad (2.10)$$

Proof. Clearly $B_{n,p,q}^{[r]}(f; x)$ is continuous on $[0, 1]$. From (1.8) we have

$$B_{n,p,q}^{[r]}(f; x) = \sum_{i=0}^r \frac{(-1)^i}{i!} B_{n,p,q}((t-x)^i f^{(i)}(t); x).$$

From the Corollary 2.5, we have

$$\begin{aligned} | B_{n,p,q}((t-x)^i f^{(i)}(t); x) | &\leq \| f^{(i)} \| B_{n,p,q}(|t-x|^i; x) \\ &\leq K_i \| f^{(i)} \| [n]^{-\frac{i}{2}}. \end{aligned}$$

Therefore

$$\begin{aligned} \| B_{n,p,q}^{[r]}(f; x) \| &\leq \sum_{i=0}^r \frac{(-1)^i}{i!} \| B_{n,p,q}((t-x)^i f^{(i)}(t); x) \| \\ &\leq M(r) \sum_{i=0}^r \| f^{(i)} \|. \end{aligned}$$

This complete the proof. \square

3. CONVERGENCE PROPERTIES OF $B_{n,p,q}^{[r]}(f; x)$

The modulus of continuity of the derivative $f^{(r)}$ is given by

$$\omega(f^{(r)}; t) = \sup \left\{ |f^{(r)}(x) - f^{(r)}(y)| : |x - y| \leq t, x, y \in [0, 1] \right\}. \quad (3.1)$$

Theorem 3.1. *Let $0 < q < p \leq 1$ and $r \in \mathbb{N} \cup \{0\}$ be a fixed number. Then for $x \in [0, 1]$, $n \in \mathbb{N}$ there exists $D_r > 0$ such that for every $f \in C^r[0, 1]$ the following inequality holds*

$$|B_{n,p,q}^{[r]}(f; x) - f(x)| \leq D_r \frac{1}{[n]^{\frac{r}{2}}} \omega\left(f^{(r)}; \frac{1}{\sqrt{[n]}}\right). \quad (3.2)$$

Proof. Let $r \in \mathbb{N}$. Then for $f \in C^r[0, 1]$ at a given point $t \in [0, 1]$, we have from the Taylor formula that

$$\begin{aligned} f(x) &= \sum_{i=0}^r \frac{f^{(i)}(t)}{i!} (x-t)^i + \frac{(x-t)^r}{((r-1)!)!} \\ &\times \int_0^1 (1-u)^{r-1} (f^{(r)}(t+u(x-t)) - f^{(r)}(t)) du. \end{aligned}$$

On applying $B_{n,p,q}^{[r]}(f; x)$, we get

$$\begin{aligned} f(x) - B_{n,p,q}^{[r]}(f; x) &= \sum_{k=0}^n \frac{(x - \frac{[k]}{p^{k-n}[n]})^r}{(r-1)!} \int_0^1 (1-u)^{r-1} P_{n,k}(p, q; x) \\ &\times \left[f^{(r)}\left(\frac{[k]}{p^{k-n}[n]} + u\left(x - \frac{[k]}{p^{k-n}[n]}\right)\right) - f^{(r)}\left(\frac{[k]}{p^{k-n}[n]}\right) \right] du. \end{aligned} \quad (3.3)$$

Now from the definition and properties of modulus of continuity, we have

$$\begin{aligned} \left| f^{(r)}\left(\frac{[k]}{p^{k-n}[n]} + u\left(x - \frac{[k]}{p^{k-n}[n]}\right)\right) - f^{(r)}\left(\frac{[k]}{p^{k-n}[n]}\right) \right| &\leq \omega\left(f^{(r)}; u\left|x - \frac{[k]}{p^{k-n}[n]}\right|\right) \\ \omega\left(f^{(r)}; u\left|x - \frac{[k]}{p^{k-n}[n]}\right|\right) &\leq \left(\sqrt{[n]}\left|x - \frac{[k]}{p^{k-n}[n]}\right| + 1\right) \omega\left(f^{(r)}; \frac{1}{\sqrt{[n]}}\right). \end{aligned} \quad (3.4)$$

□

Now for every $0 \leq x \leq 1$, $0 < q < p \leq 1$, $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ and from (3.3) and (3.4), we get

$$\begin{aligned} &|B_{n,p,q}^{[r]}(f; x) - f(x)| \\ &\leq \frac{1}{r!} \omega\left(f^{(r)}; \frac{1}{\sqrt{[n]}}\right) \sum_{k=0}^n \left|x - \frac{[k]}{p^{k-n}[n]}\right|^r \left(\sqrt{[n]}\left|x - \frac{[k]}{p^{k-n}[n]}\right| + 1\right) P_{n,k}(p, q; x) \end{aligned}$$

$$= \frac{1}{r!} \omega \left(f^{(r)}; \frac{1}{\sqrt{[n]}} \right) \left(\sqrt{[n]} B_{n,p,q}(|x-t|^{r+1}; x) + B_{n,p,q}(|x-t|^r; x) \right). \quad (3.5)$$

Using (3.9) and (3.5) for $x \in [0, 1]$, we have

$$\begin{aligned} |B_{n,p,q}^{[r]}(f; x) - f(x)| &\leq \frac{1}{r!} (K_{r+1} + K_r) \left(\frac{1}{\sqrt{[n]}} \right)^r \omega \left(f^{(r)}; \frac{1}{\sqrt{[n]}} \right) \\ &= D_r \left(\frac{1}{\sqrt{[n]}} \right)^r \omega \left(f^{(r)}; \frac{1}{\sqrt{[n]}} \right). \end{aligned}$$

In order to obtain the uniform convergence of $B_{n,p_n,q_n}^{[r]}(f; x)$ to a continuous function f , we take $q = q_n$, $p = p_n$ where $q_n \in (0, 1)$ and $p_n \in (q_n, 1]$ satisfying,

$$\lim_n p_n = 1, \quad \lim_n q_n = 1. \quad (3.6)$$

Corollary 3.2. *Let $p = p_n$, $q = q_n$, $0 < q_n < p_n \leq 1$ satisfy (3.6) and $f \in C^r[0, 1]$ for a fixed number $r \in \mathbb{N} \cup \{0\}$. Then*

$$\lim_{n \rightarrow \infty} [n]^{\frac{r}{2}} \|B_{n,k}^{[r]}(f) - f\| = 0. \quad (3.7)$$

We say that (cf. [13]) a function $f \in C[0, 1]$ belongs to $Lip_M(\alpha)$, $0 < \alpha \leq 1$, provided

$$|f(x) - f(y)| \leq M |x - y|^\alpha, \quad (x, y \in [0, 1] \text{ and } M > 0). \quad (3.8)$$

Corollary 3.3. *Let $p = p_n$, $q = q_n$, $0 < q_n < p_n \leq 1$ satisfy (3.6) and $f \in C^r[0, 1]$ for a fixed number $r \in \mathbb{N} \cup \{0\}$. If $f^{(r)} \in Lip_M(\alpha)$ then*

$$\|B_{n,p,q}^{[r]}(f) - f\| = O\left([n]^{-\frac{r+\alpha}{2}}\right). \quad (3.9)$$

Proof. From (3.2) and (3.8), we have

$$\|B_{n,p,q}^{[r]}(f) - f\| \leq D_r M \frac{1}{[n]^{\frac{r}{2}}} \frac{1}{[n]^{\frac{\alpha}{2}}}.$$

□

Theorem 3.4. *Let $0 < q < p \leq 1$. Suppose that $f \in C^{r+2}[0, 1]$, where $r \in \mathbb{N} \cup \{0\}$ is fixed then we have*

$$\begin{aligned} &\left| B_{n,p,q}^{[r]}(f; x) - f(x) - \frac{(-1)^r f^{(r+1)}(x) B_{n,p,q}((t-x)^{r+1}; x)}{(r+1)!} \right. \\ &\quad \left. - \frac{(-1)^r f^{(r+2)}(x) B_{n,p,q}((t-x)^{r+2}; x)}{(r+2)!} \right| \\ &\leq (K_{r+2} + K_{r+4}) \frac{x(1-x)}{[n]^{\frac{r}{2}+1}} \sum_{i=0}^r \frac{1}{i!(r+2-i)!} \omega \left(f^{(r+2-i)}, [n]^{-\frac{1}{2}} \right). \end{aligned}$$

Proof. Let $f \in C^{r+2}[0, 1]$ and $x \in [0, 1]$ for a fixed number $r \in \mathbb{N} \cup \{0\}$ we have $f^{(i)} \in C^{r+2-i}[0, 1]$, $0 \leq i \leq r$. Then by Taylor formula we can write

$$f^{(i)}(t) = \sum_{j=0}^{r+2-i} \frac{f^{(i+j)}(x)}{j!} (t-x)^j + R_{r+2-j}(f; t; x), \quad (3.10)$$

where

$$R_{r+2-i}(f; t; x) = \frac{f^{(r+2-i)}(\zeta_{p^{k-n}t}) - f^{(r+2-i)}(x)}{(r+2-i)!} (t-x)^{r+2-i},$$

and

$$|\zeta_t - x| < |t - x|.$$

Therefore from (1.8) and (3.10) we have

$$\begin{aligned} B_{n,p,q}^{[r]}(f; x) &= \sum_{k=0}^n P_{n,k}(p, q; x) \sum_{i=0}^r \frac{\left(x - \frac{[k]}{p^{k-n}[n]}\right)^i}{i!} \sum_{j=0}^{r+2-i} \frac{f^{(i+j)}(x)}{j!} \left(\frac{[k]}{p^{k-n}[n]} - x\right)^j \\ &+ \sum_{k=0}^n P_{n,k}(p, q; x) \sum_{i=0}^r \frac{\left(x - \frac{[k]}{p^{k-n}[n]}\right)^i}{i!} R_{r+2-i}(f; t; x) \\ &= I_1 + I_2, \text{ \{where } t = \frac{[k]}{p^{k-n}[n]}\text{ \}} \end{aligned}$$

Which implies that

$$\begin{aligned} &|B_{n,p,q}^{[r]}(f; x) - I_1| \\ &= |I_2| \\ &= \left| \sum_{k=0}^n P_{n,k}(p, q; x) \sum_{i=0}^r \frac{(-1)^i}{i!} \frac{f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x)}{(r+2-i)!} (t-x)^{r+2} \right| \\ &= \left| B_{n,p,q} \left(\sum_{i=0}^r \frac{(-1)^i}{i!} \frac{f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x)}{(r+2-i)!} (t-x)^{r+2}; x \right) \right|. \end{aligned}$$

We use the well-known inequality

$$\omega(f, \lambda\delta) \leq (1 + \lambda^2)\omega(f, \delta),$$

$$\begin{aligned} |f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x)| &\leq \omega(f^{(r+2-i)}, |\zeta_t - x|) \\ &\leq \omega(f^{(r+2-i)}, |t - x|) \\ &\leq \omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right) (1 + [n](t-x)^2). \end{aligned}$$

Hence

$$\begin{aligned} |I_2| &\leq \left| B_{n,p,q} \left(\sum_{i=0}^r \frac{(-1)^i}{i!} \frac{f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x)}{(r+2-i)!} \right) |t-x|^{r+2}; x \right| \\ &\leq B_{n,p,q} \left(\sum_{i=0}^r \frac{1}{i!(r+2-i)!} \omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right) (1 + [n](t-x)^2) |t-x|^{r+2}; x \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=0}^r \frac{1}{i!(r+2-i)!} \omega \left(f^{(r+2-i)}, [n]^{-\frac{1}{2}} \right) \\
 &\times (B_{n,p,q}(|t-x|^{r+2}; x) + [n]B_{n,p,q}(|t-x|^{r+4}; x)) \\
 &\leq \sum_{i=0}^r \frac{1}{i!(r+2-i)!} \omega \left(f^{(r+2-i)}, [n]^{-\frac{1}{2}} \right) \left(K_{r+2} \frac{x(1-x)}{[n]^{\frac{r}{2}+1}} + K_{r+4} \frac{x(1-x)}{[n]^{\frac{r}{2}+1}} \right) \\
 &= (K_{r+2} + K_{r+4}) \frac{x(1-x)}{[n]^{\frac{r}{2}+1}} \sum_{i=0}^r \frac{1}{i!(r+2-i)!} \omega \left(f^{(r+2-i)}, [n]^{-\frac{1}{2}} \right).
 \end{aligned}$$

Therefore

$$|B_{n,p,q}^{[r]}(f; x) - I_1| \leq (K_{r+2} + K_{r+4}) \frac{x(1-x)}{[n]^{\frac{r}{2}+1}} \sum_{i=0}^r \frac{1}{i!(r+2-i)!} \omega \left(f^{(r+2-i)}, [n]^{-\frac{1}{2}} \right).$$

Now we simplify for I_1

$$\begin{aligned}
 I_1 &= \sum_{k=0}^n P_{n,k}(p, q; x) \sum_{i=0}^r \frac{(x - \frac{[k]}{p^{k-n}[n]})^i}{i!} \sum_{l=i}^{r+2} \frac{f^{(l)}(x)}{(l-i)!} \left(\frac{[k]}{p^{k-n}[n]} - x \right)^{l-i} \\
 &= \sum_{k=0}^n P_{n,k}(p, q; x) \sum_{i=0}^r \frac{(-1)^i}{i!} \sum_{l=i}^r \frac{f^{(l)}(x)}{(l-i)!} \left(\frac{[k]}{p^{k-n}[n]} - x \right)^l \\
 &+ \sum_{k=0}^n P_{n,k}(p, q; x) \sum_{i=0}^r \frac{(-1)^i}{i!} \frac{f^{(r+1)}(x)}{(r+1-i)!} \left(\frac{[k]}{p^{k-n}[n]} - x \right)^{r+1} \\
 &+ \sum_{k=0}^n P_{n,k}(p, q; x) \sum_{i=0}^r \frac{(-1)^i}{i!} \frac{f^{(r+2)}(x)}{(r+2-i)!} \left(\frac{[k]}{p^{k-n}[n]} - x \right)^{r+2} \\
 &= \sum_{k=0}^n P_{n,k}(p, q; x) \sum_{l=0}^r \frac{f^{(l)}(x)}{(l)!} \left(\frac{[k]}{p^{k-n}[n]} - x \right)^l \sum_{i=0}^l \binom{l}{i} (-1)^i \\
 &+ \frac{f^{(r+1)}(x)}{(r+1)!} \sum_{k=0}^n P_{n,k}(p, q; x) \left(\frac{[k]}{p^{k-n}[n]} - x \right)^{r+1} \sum_{i=0}^r \binom{r+1}{i} (-1)^i \\
 &+ \frac{f^{(r+2)}(x)}{(r+2)!} \sum_{k=0}^n P_{n,k}(p, q; x) \left(\frac{[k]}{p^{k-n}[n]} - x \right)^{r+2} \sum_{i=0}^r \binom{r+2}{i} (-1)^i.
 \end{aligned}$$

For $n \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$ we have

$$\sum_{i=0}^r \binom{r+1}{i} (-1)^i = (-1)^r, \quad \sum_{i=0}^r \binom{r+2}{i} (-1)^i = (r+1)(-1)^r$$

Therefore

$$\begin{aligned}
 I_1 &= f(x) + \frac{(-1)^r f^{(r+1)}(x) B_{n,p,q}((t-x)^{r+1}; x)}{(r+1)!} \\
 &+ \frac{(-1)^r f^{(r+2)}(x) B_{n,p,q}((t-x)^{r+2}; x)}{(r+2)!}.
 \end{aligned}$$

This complete the proof. □

Corollary 3.5. *Let $p = p_n$, $q = q_n$, $0 < q_n < p_n \leq 1$ satisfy (3.6) and $f \in C^2[0, 1]$ for a fixed number $r \in \mathbb{N} \cup \{0\}$. Then for every $x \in [0, 1]$ we have*

$$\left| B_{n,p_n,q_n}^{[r]}(f; x) - f(x) - \frac{f''(x)}{2} \frac{x(1-x)}{[n]} \right| \leq K \frac{x(1-x)}{[n]} \omega\left(f'', [n]^{-\frac{1}{2}}\right),$$

where $K = \frac{K_2+K_4}{2}$. Moreover,

$$\lim_{n \rightarrow \infty} [n] (B_{n,p_n,q_n}(f; x) - f(x)) = \frac{x(1-x)}{2} f''(x)$$

uniformly on $[0, 1]$.

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